Microwave Transmission System General representation of EM field in terms of TEM, TE and TM components, rectangular wave guides, Circular Wave guides, Solution in terms of various modes, Properties of propagating modes, Dominant modes, effect of higher order modes, Strip line and micro strip lines general properties, Comparison of coaxial, Micro strip and rectangular wave guides in terms of band width, power handling capacity, economical consideration etc.

MICROWAVE ENGINEERING

Maxwell's equation:

I. $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ **(Ampere's law)**

- $\vec{H}$: Magnetic field Intensity
- $\vec{J}$: Conduction current density
- $\vec{D}$: Displacement current density

Statement:

It gives induced magnetic field in a closed circuit due to conduction current density i.e. constant current throughout conduction and displacement current density due to time varying current components.

II. Faraday's Law:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- $\vec{E}$: Electric field intensity
- $\vec{B}$: Magnetic flux density

Statement:

It gives EMF induced in closed circuit due to change of magnetic flux.

WAVE EQUATION

$$\vec{E} = E_o e^{j\omega t} \qquad (1)$$

Diff. (1) ω.r.t

$$\frac{\partial \vec{E}}{\partial t} = E_o e^{j\omega t}.\,(j\omega) \qquad (2)$$

$$\frac{\partial \vec{E}}{\partial t} = \vec{E}.\,(j\omega)$$

$$\frac{\partial}{\partial t} = j\omega$$

Diff. (2) w.r.t. t :

$$\frac{\partial^2 \vec{E}}{\partial t^2} = E_o e^{j\omega t}.\,(j\omega)^2$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = \vec{E}\,(-\omega^2)$$

$$\frac{\partial^2}{\partial t^2} = -\omega^2$$

From Maxwell I equation

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial \vec{t}} = \sigma \vec{E} + \sum \frac{\partial \vec{E}}{\partial t}$$

For air, $\sigma = \rho = 0$

So, $\nabla \times \vec{H} = \sum \frac{\partial \vec{E}}{\partial t}$

$$\nabla \times \vec{H} = \frac{d\vec{E}}{dt}$$

$$\nabla \times \vec{H} = j\omega \sum \vec{E} \qquad (3)$$

From Maxwell's II equation:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Since, $B = \mu \vec{H}$

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t} \mu \vec{H}$$

$$\nabla \times \vec{E} = -\omega j \mu \vec{H}$$

Taking curl of (3) equation:

$$\nabla \times \nabla \times \vec{H} = j\omega \sum (\nabla \times \vec{E})$$

$$= j\omega \sum (-j\omega\mu\vec{H})$$

$$\nabla \times \nabla \times \vec{H} = \omega^2 \mu \sum \vec{H}$$

$$\nabla \times \nabla \times \vec{H} = \nabla(\nabla.\vec{H}).\nabla^2 \vec{H}$$

By Maxwell's IV equation:

$$\nabla \times \vec{B} = 0$$
$$B = \mu H$$
$$\mu \nabla.\vec{H} = 0$$
$$\mu \neq 0$$
$$\nabla \times \vec{H} = 0$$

$\{C^2 = 1/\mu E\}$

So,

$$\nabla \times \nabla \times \vec{H} = \nabla^2 \vec{H}$$
$$So, \nabla^2 \vec{H} = -\omega^2 \mu \sum \vec{H}$$

Similarly,

$$\nabla^{\vec{2}} \vec{E} = -\omega^2 \mu \sum \vec{E}$$

Replace $-\omega^2 by \frac{\partial^2}{\partial t^2}$

$$So, \nabla^2 \vec{H} = \frac{1}{C^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$Similarly, \nabla^2 \vec{E} = \frac{1}{C^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

TEM/TE/TM/HE WAVE EQUATION

TEM:-_ $H_2 = 0$ Ez=0

TE: $H_2 \neq 0$ Ez $\neq 0$

TM: $H_2 = 0$ Ez $\neq 0$

HE: $H_2 \neq 0$ Ez $\neq 0$

TRANSMISSION LINES

- Transmission lines are limited to high frequency application.
- Length of transmission lines α Magnitude and wavelength of signal.
- Types of transmission lines:

1. 2 wire parallel transmission lines
2. Coaxial lines
3. Strip type substrate transmission lines
4. Waveguides

2 WIRE PARALLEL TRANSMISSION LINES

- Pair of uniform size wires used for transmission of electrical energy.
- Uses: Power transmission, telephone lines and T.V signals.
- Frequency: Below 500 MHz (Due to radiation losses)
- Non perfect dielectric.

WAVEGUIDES

Microwave transmission lines.

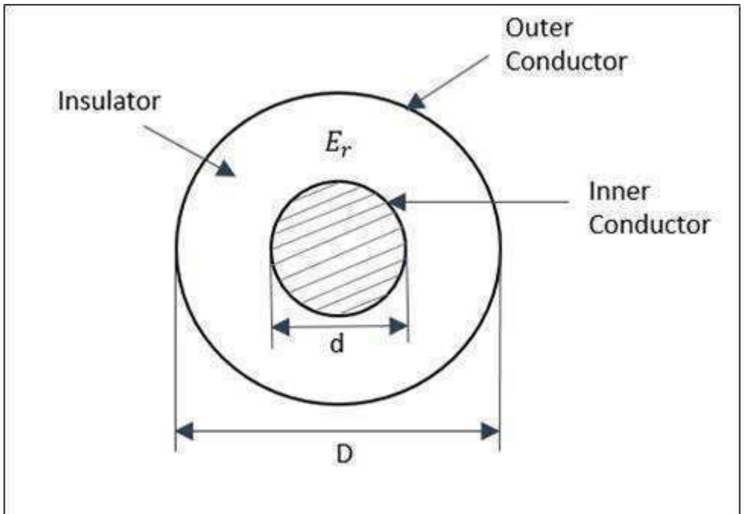
All microwave frequency following transmission lines will be employed:

1) Multi-conduction lines- Support TEM.
- Coaxial lines
- Microstrip lines
- Strip lones
- Slot lines

2) Single conductor lines (Waveguides)- TE/TM mode and HE mode.
- Rectangular waveguide
- Circular waveguide

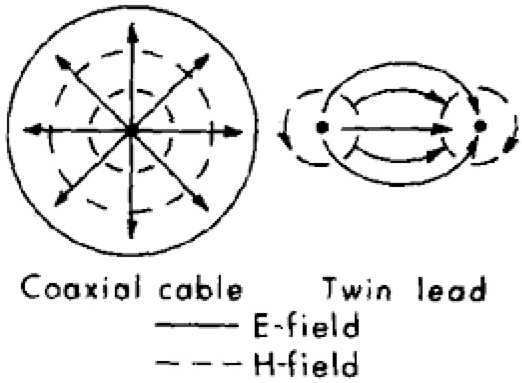
Multiconductor transmission lines

1) Coaxial lines
- Use- High frequency application
- Support TEM mode.



Cross-Sectional View of a Co-axial line

TEM wave in Coaxial lines



$\mu_r = 1$

$Ev \neq 1$

$\mu_0 = 4\pi \times 10^{-7} H/m$

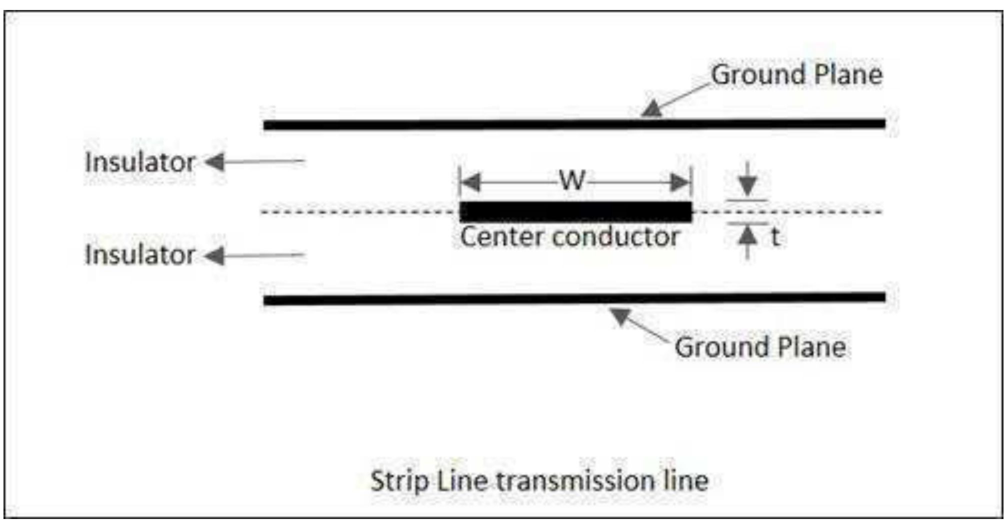
$\varepsilon = \varepsilon_0 \varepsilon_v$

$\varepsilon_0 = 8.854 \times 10^{-12} F/m$

2) Strip lines:

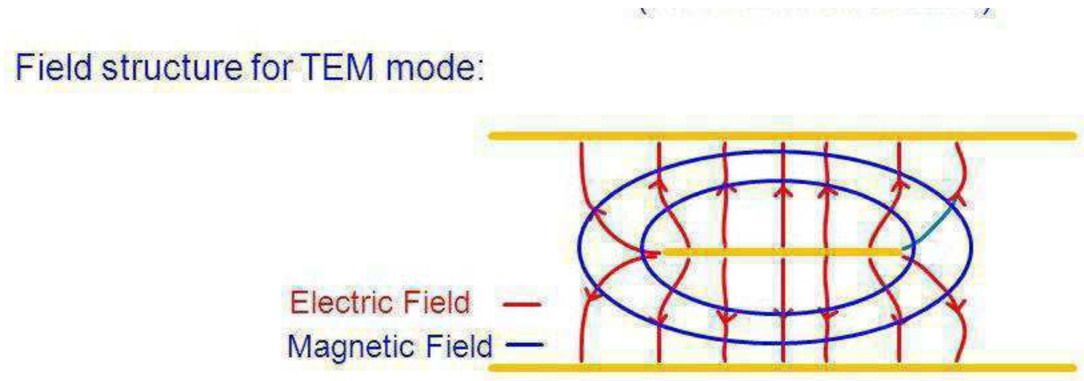
Planar transmission lines.

Frequency range: 100MHz to 100GHz



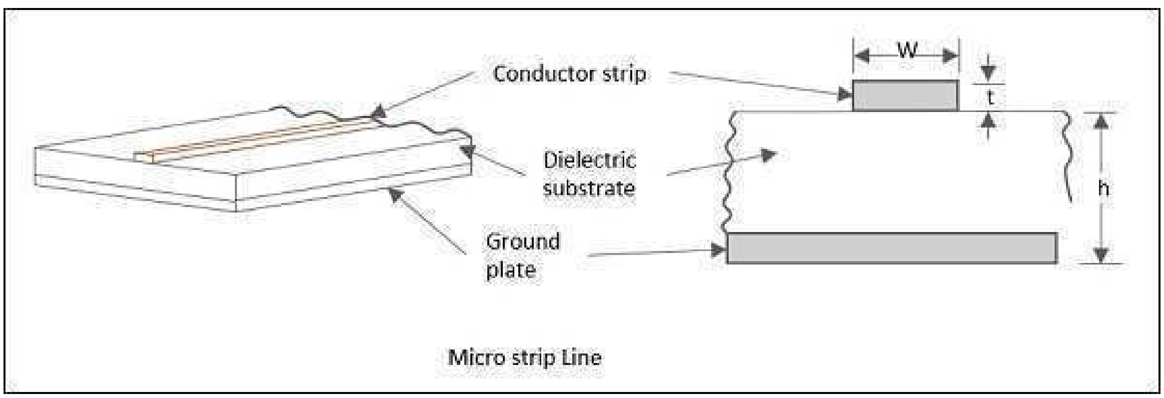
Mode: TEM

Field confined within transmission lines with no radiation losses.



3) Microstrip lines:

Unsymmetrical strip lines or parallel plate strip lines having di-electric substrate which has a metallized ground on the bottom and a thin conducting strip on top with thickness '**t**' and width '**ω**'.



Micro strip Line

Advantages of micro strip over strip lines, co-axial and waveguides.

1) Complete conductor pattern may be deposited and processed on a single dielectric substrate which is supported by a single metal ground plane. So, fabrication cost would be substantially lower than stripline, coaxial and waveguide.
2) Due to planar nature of microstrip structures both packaged and unpackaged semiconductor chips can be conveniently attached to microstrip elements.
3) There is easy way to access top surface making it easy to mount passive or active discrete components and also for making minor adjustments.

LIMITATIONS:

1) Due to open microstrip structure, high radiation losses or interference due to nearby conductors.
2) Due to air-dielectric-air interface discontinuity in electric and magnetic field is generated.

Frequency bands: 1 GHz, 5 GHz and 10 GHz.

Frequency range: 0-50 GHz.

Microstrip antenna- Use PCB.

WAVEGUIDE (Single lines)

- Frequency range is higher than 3 GHz.
- Hollow metallic tube called waveguide.
- Used in UHF and Microwave regions

Comparison of waveguide with 2- wire transmission lines

Similarities-

1) Wave travelling in a waveguide has a phase velocity abd will be attenuated as in a transmission lines.
2) When the wane reaches the end of waveguide, it is reflected unless the load impedence is adjusted to absorb the wave.
3) Any irregularity in a waveguide produces reflections just like irregularities as in transmission lines.
4) Reflected wave can be eliminated by proper impedence matching as in transmission line.
5) When both incident and reflected waves are present in a waveguide, a standing wave pattern results as in a transmission line.

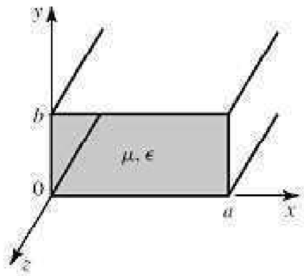
Dissimilarities:

1) Waveguide as H.P.F with cut off frequency f_c. In 2-wire lossless transmission lines, all frequency can pass through.

2) Waveguide one conductor transmission system. Body of waveguide acts as ground and wave propagates through multiple reflections from walls of waveguide.
3) Propagation in waveguide in accordance to field theory. Transmission line propagation in accordance to circuit theory.

PROPAGATION OF WAVES IN RECTANGULAR WAVEGUIDE

Consider a waveguide situated in rectangular coordinate system with its breadth along x-axis and width along y-axis and propagation along z-axis, with air as dielectric medium.



Geometry of a rectangular waveguide

$$\nabla^2 \overrightarrow{H}^2 = -\omega^2 \mu\varepsilon \overrightarrow{H}z \text{ for TE wave (Ez=0)}$$

$$\nabla^2 \vec{\mathrm{E}}\mathrm{z} = -\omega^2 \mu\varepsilon \vec{E}z \text{ for TM wave (Hz=0)}$$

Expanding $\nabla^2 \vec{\mathrm{E}}\mathrm{z}$ in rectangular coordinate system:

$$\frac{\partial^2 \vec{\mathrm{E}}\mathrm{z}}{\partial x^2} + \frac{\partial^2 \vec{\mathrm{E}}\mathrm{z}}{\partial y^2} + \frac{\partial^2 \vec{\mathrm{E}}\mathrm{z}}{\partial z^2} = -\omega^2 \mu\varepsilon \vec{E}z$$

$$\frac{\partial^2}{\partial z^2} = x^2$$

$$\frac{\partial^2 \vec{\mathrm{E}}\mathrm{z}}{\partial x^2} + \frac{\partial^2 \vec{\mathrm{E}}\mathrm{z}}{\partial y^2} + x^2 \vec{\mathrm{E}}\mathrm{z} = -\omega^2 \mu\varepsilon \vec{E}z$$

$$\frac{\partial^2 \vec{\mathrm{E}}\mathrm{z}}{\partial x^2} + \frac{\partial^2 \vec{\mathrm{E}}\mathrm{z}}{\partial y^2} + (x^2 + \omega^2 \mu\varepsilon)\vec{E}z = 0$$

$$Let\ \mathrm{h}^2 = x^2 + \omega^2 \mu\varepsilon$$

$$\frac{\partial^2 \vec{\mathrm{E}}\mathrm{z}}{\partial x^2} + \frac{\partial^2 \vec{\mathrm{E}}\mathrm{z}}{\partial y^2} + \mathrm{h}^2 Ez = 0 \text{ (For TM wave)} \qquad (1)$$

Similarly for Hz:

$$\frac{\partial^2 \overrightarrow{Hz}}{\partial x^2} + \frac{\partial^2 \overrightarrow{\mathrm{Hz}}}{\partial y^2} + \mathrm{h}^2 Hz = 0 \text{ (For TE wave)} \qquad (2)$$

By solving above (1) and (2) equation, we get solution for Ez and Hz.

Now using Maxwell's equation for finding the various components $(\overrightarrow{Ex}, \overrightarrow{Ey}, \overrightarrow{Hx}, Hy)$:

From Maxwell's I equation:

$$\nabla \times \overrightarrow{H} = j\omega\varepsilon \overrightarrow{E}$$

Expanding above equation

$$\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \overrightarrow{Hx} & \overrightarrow{Hy} & \overrightarrow{Hz} \end{pmatrix} = j\omega\varepsilon \left[\hat{i}\,\overrightarrow{Ex} + \hat{j}\,\overrightarrow{Ey} + \hat{k}\,\overrightarrow{Ez} \right]$$

$$\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & -x \\ \overrightarrow{Hx} & \overrightarrow{Hy} & \overrightarrow{Hz} \end{pmatrix} = j\omega\varepsilon \left[\hat{i}\,\overrightarrow{Ex} + \hat{j}\,\overrightarrow{Ey} + \hat{k}\,\overrightarrow{Ez} \right]$$

$$\frac{\partial}{\partial y} \overrightarrow{Hz} + x.\overrightarrow{Hy} = -j\omega\varepsilon \overrightarrow{Ex} \qquad (3)$$

$$\frac{\partial}{\partial x} \overrightarrow{Hz} + x.\overrightarrow{Hx} = -j\omega\varepsilon \overrightarrow{Ey} \qquad (4)$$

$$\frac{\partial}{\partial x} \overrightarrow{Hy} - \frac{\partial}{\partial y} \overrightarrow{Hx} = j\omega\varepsilon \overrightarrow{Ez} \qquad (5)$$

Maxwell's 2nd equation

$$\nabla \times \overrightarrow{E} = -j\omega\mu \overrightarrow{H}$$

$$\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \overrightarrow{Ex} & \overrightarrow{Ey} & \overrightarrow{Ez} \end{pmatrix} = j\omega\mu \left[\hat{i}\,\overrightarrow{Hx} + \hat{j}\,\overrightarrow{Hy} + \hat{k}\,\overrightarrow{Hz} \right]$$

$$\frac{\partial}{\partial y}\overrightarrow{Ez} + x.\overrightarrow{Ey} = -j\omega\mu\overrightarrow{Hx} \qquad (6)$$

$$\frac{\partial}{\partial x}\overrightarrow{Ez} + x.\overrightarrow{Ex} = +j\omega\mu\overrightarrow{Hy} \qquad (7)$$

$$\frac{\partial}{\partial x}\overrightarrow{Ey} - \frac{\partial}{\partial y}\overrightarrow{Ex} = -j\omega\mu\overrightarrow{Hz} \qquad (8)$$

From (3) and (7) eliminate $\overline{\text{Hy}}$ and get $\overrightarrow{Ex}$

From (7):

$$\frac{1}{j\omega\mu}\frac{\partial}{\partial x}\overrightarrow{Ez} + \frac{x}{j\omega\mu}\overrightarrow{Ex} = \overrightarrow{Hy}$$

Put $\overrightarrow{Hy}$ in (3)

$$\frac{\partial}{\partial y}\overrightarrow{Hz} + x.\left[\frac{1}{j\omega\mu}\frac{\partial}{\partial x}\overrightarrow{Ez} + \frac{x}{j\omega\mu}\overrightarrow{Ex}\right] = j\omega\varepsilon\overrightarrow{Ex}$$

$$\frac{\partial}{\partial y}\overrightarrow{Hz} + \left[\frac{x}{j\omega\mu}\frac{\partial}{\partial x}\overrightarrow{Ez} + \frac{x^2}{j\omega\mu}\overrightarrow{Ex}\right] = j\omega\varepsilon\overrightarrow{Ex}$$

$$\frac{\partial}{\partial y}\overrightarrow{Hz} + \frac{x}{j\omega\mu}\frac{\partial}{\partial x}\overrightarrow{Ez} = \left[j\omega\varepsilon - \frac{x^2}{j\omega\mu}\right]\overrightarrow{Ex}$$

$$\frac{\partial}{\partial y}\overrightarrow{Hz} + \frac{x}{j\omega\mu}\frac{\partial}{\partial x}\overrightarrow{Ez} = \left[\frac{-\omega^2\mu\varepsilon x^2}{j\omega\mu}\right]\overrightarrow{Ex}$$

$$\frac{\partial}{\partial y}\overrightarrow{Hz} + \frac{x}{j\omega\mu}\frac{\partial}{\partial x}\overrightarrow{Ez} = \frac{-h^2}{j\omega\mu}\overrightarrow{Ex}$$

Multiply $j\omega\mu$ both sides

Multiply $j\omega\mu$ both sides

$$-j\omega\mu\frac{\partial}{\partial y}\overrightarrow{Hz} - x\frac{\partial}{\partial x}\overrightarrow{Ez} = h^2\overrightarrow{Ex}$$

$$\boxed{\overrightarrow{Ex} = -\frac{j\omega\mu}{h^2}\frac{\partial}{\partial y}\overrightarrow{Hz} - \frac{x}{h^2}\frac{\partial}{\partial x}\overrightarrow{Ez}}$$

$$\overrightarrow{Ex} = -\frac{x}{h^2}\frac{\partial}{\partial x}\overrightarrow{Ez} - \frac{j\omega\mu}{h^2}\frac{\partial}{\partial y}\overrightarrow{Hz} \qquad (9)$$

$$\overrightarrow{Ey} = -\frac{x}{h^2}\frac{\partial}{\partial x}\overrightarrow{Ez} + \frac{j\omega\mu}{h^2}\frac{\partial}{\partial x}\overrightarrow{Hz} \qquad (10)$$

$$\overrightarrow{Hx} = -\frac{x}{h^2}\frac{\partial}{\partial x}\overrightarrow{Hz} + \frac{j\omega\varepsilon}{h^2}\frac{\partial}{\partial x}\overrightarrow{Ez} \qquad (11)$$

$$\overrightarrow{Hy} = -\frac{x}{h^2}\frac{\partial}{\partial x}\overrightarrow{Hz} - \frac{j\omega\varepsilon}{h^2}\frac{\partial}{\partial x}\overrightarrow{Ez} \qquad (12)$$

TE and TM Modes

- The EMW inside a waveguide can have infinite number of patterns which are called modes. Fields in the waveguide which make up mode pattern must obey certain physical laws.
- At surface of conductor:

  Electric field is always perpendicular to conductor.

  Magnetic field is always parallel to conductor.

- Two modes in rectangular waveguides:
  1. TE
  2. TM

Field patterns:- Fig.1 shows field pattern for a TE wave, solid line for electric field lines or voltage lines and dotted lines depicts magnetic field.

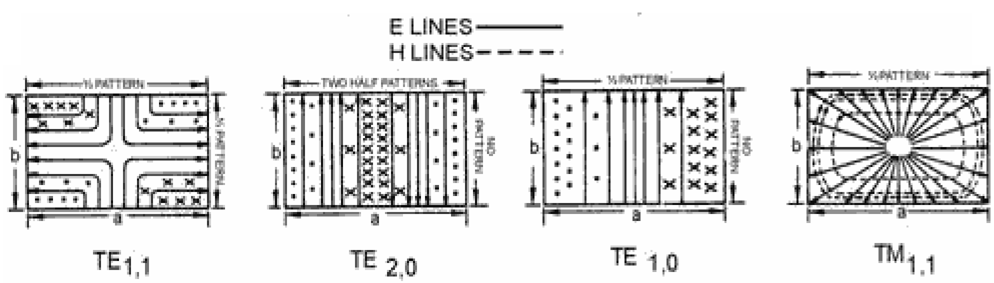


Fig.1

We use subscript for designating particular mode. TE_{mn} or TM_{mn} where,

m : Indicates number of half wave variations of electric field across wide dimension ‘a’ of waveguide or magnetic field for TM mode.

n : indicates number of half wave variations of electric field across narrow dimension ‘b’.

Referring to TE pattern shown in Fig. 1,it can be seen that voltage varies from 0 to max. and max. to 0 across wider dimension ‘a’. This is one half variations in volt V so m=1. In narrow dimension there is no variation so n=0.

Therefore, this is called TE_{10} mode. The mode having highest cut-off wavelength is known as Dominant mode of waveguide and all other modes are high order modes.

TE_{10} mode is dominant mode for TE waves. It is the mode which is used for practically all electromagnetic transmission in a rectangular waveguide.

DOMINANT MODE- Low loss, distortionless transmission.

HIGHER MODE- Significant power loss and undesirable harmonic distortion.

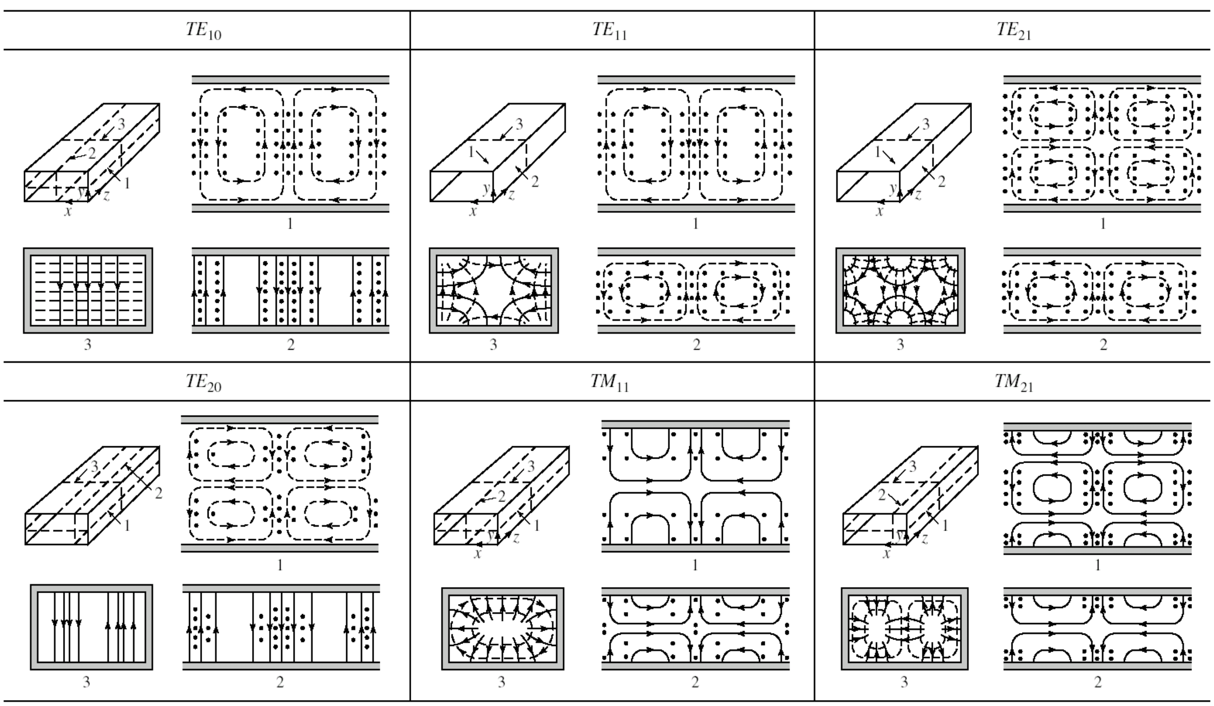
OTHER HIGH ORDER MODES:

TE_{01} mode

TE_{20} mode

TE_{30} mode

TE_{11} mode



PROPAGATION OF TM MODE IN RECTANGULAR WAVEGUIDE

$Ez \neq 0, Hz = 0$ (For TM wave)

Wave equation for TM wave:

$$\frac{\partial^2 \overrightarrow{Ez}}{\partial x^2} + \frac{\partial^2 \overrightarrow{Ez}}{\partial y^2} + h^2 \overrightarrow{Ez} = 0$$

This is p.d.e which can be solved to get different field components Ex, Ey, Hx, Hy by separation of variable method

Let a solution:

$$Ez = XY \qquad (1)$$

X is pure function of *x*.

Y is pure function of y.

$$\frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2} + h^2 XY = 0$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + h^2 XY = 0 \qquad (2)$$

Divide XY in equation (2)

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + h^2 Y = 0 \qquad (3)$$

$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}$ is pure function of *x*.

$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$ is pure function of y.

Sum of these are constant.

Using separation of variable method to solve the differential equation (3)

Let $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -B^2 \qquad (4)$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -A^2 \qquad (5)$$

So, $-A^2 - B^2 + h^2 = 0$

$$\boxed{h^2 = A^2 + B^2}$$

Equation (4) and (5) are II order difference equation so solution of which are

$$X = C_1 \cos Bx + C_2 \sin Bx$$

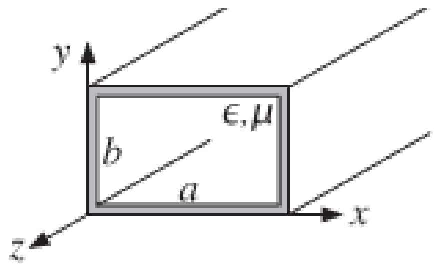
$$Y = C_3 \cos Ay + C_4 \sin Ay$$

Now,

$$Ez = XY$$

$$Ez = (C_1 \cos Bx + C_2 \sin Bx)(C_3 \cos Ay + C_4 \sin Ay) \qquad (6)$$

BOUNDARY CONDITIONS:



I Boundary Condition: (Bottom Plane or Bottom Wall) i.e.

$\overrightarrow{Ez} = 0$ y=0 ƛ $x \rightarrow 0$ to a

II Boundary Condition: (LHS plane)

$\overrightarrow{Ez} = 0 \; x = 0 \quad y \rightarrow 0$ to b

III Boundary Condition: (Top plane)

$\overrightarrow{Ez} = 0 \; y = b \quad x \rightarrow 0 \; to \; a$

IV Boundary Condition: (RHS plane)

$\overrightarrow{Ez} = 0 \; x = a \quad y \rightarrow 0$ to b

Applying I Boundary condition in (6) equation:

$(C_1 \cos Bx + C_2 \sin Bx) C_3 = 0$

$(C_1 \cos Bx + C_2 \sin Bx) \neq 0 \qquad \boxed{C_3 = 0}$

$$Ez = (C_1 \cos Bx + C_2 \sin Bx) C_4 \sin Ay \qquad (7)$$

Applying II Boundary condition in equation (7)

$C_1 C_4 \sin Ay = 0 \qquad \sin Ay \neq 0$ and $C_4 \neq 0$

$C_1 = 0$

$$So, Ez = C_2 C_4 \sin Bx \sin Ay \qquad (8)$$

Applying III Boundary Condition in equation (8)

$$C_2 C_4 \sin Bx \sin Ab = 0$$
$$C_2 \neq 0 \ C_4 \neq 0 \ \sin Bx \neq 0$$
$$\sin Ab = 0$$
$$Ab = n\pi$$
$$\boxed{A = \frac{n\pi}{b}}$$

*Now IV Boundary Condition in equation*8 :

$$C_2 C_4 \sin Ba \sin Ay = 0$$

Since,

$$\sin Ay \neq 0, C_2 \neq 0 \ C_4 \neq 0$$
$$Now, \sin Ba = 0$$
$$Ba = m\pi$$
$$\boxed{B = \frac{m\pi}{a}}$$

Put A and B in equation (8)

$$Ez = C_2 C_4 \sin\left[\frac{m\pi}{a}\right] x \sin\left[\frac{n\pi}{b}\right] y.e^{+j\omega t}.\mathrm{e}^{-xz}$$

e^{-xz} : proportion along Z direction

$e^{+j\omega t}$: Sinusoidal variation

Ez is known . $Ex, Ey, Hx, Hy.$

$$C = C_2 C_4$$
$$Ez = C \sin\left[\frac{m\pi}{a}\right] x \sin\left[\frac{n\pi}{b}\right] y.e^{+j\omega t - xz}$$

$$\overrightarrow{Ex} = \frac{-x}{h^2}\frac{\partial \overrightarrow{Ez}}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial \overrightarrow{Hz}}{\partial y}$$

$$\overrightarrow{Ex} = \frac{-x}{h^2}\frac{\partial}{\partial x}\left[C\sin\left[\frac{m\pi}{a}\right]x\sin\left[\frac{n\pi}{b}\right]y.e^{+j\omega t-xz}\right]$$

$$\boxed{\overrightarrow{Ex} = \frac{-x}{h^2}\left[C\left[\frac{m\pi}{a}\right]\cos\left[\frac{m\pi}{a}\right]x\sin\left[\frac{n\pi}{b}\right]y.e^{(+j\omega t-xz)}\right]}$$

$$\overrightarrow{Ey} = \frac{-x}{h^2}\frac{\partial \overrightarrow{Ez}}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial \overrightarrow{Hz}}{\partial x} \qquad (\overrightarrow{Hz}=0)$$

$$\overrightarrow{Ey} = \frac{-x}{h^2}\left[\frac{\partial}{\partial y}C\sin\left[\frac{m\pi}{a}\right]x\sin\left[\frac{n\pi}{b}\right]y.e^{(+j\omega t-xz)}\right]$$

$$\boxed{\overrightarrow{Ey} = \frac{-x}{h^2}C\left[\frac{n\pi}{b}\right]\cos\left[\frac{n\pi}{b}\right]y\ \sin\left[\frac{m\pi}{a}\right]xe^{(+j\omega t-xz)}}$$

$$\overrightarrow{Hx} = \frac{-x}{h^2}\frac{\partial \overrightarrow{Hz}}{\partial x} + \frac{j\omega\mu}{h^2}\frac{\partial \overrightarrow{Ez}}{\partial y} \qquad (\overrightarrow{Hz}=0)$$

$$\boxed{\overrightarrow{Hx} = \frac{j\omega\mu}{h^2}C\left[\frac{n\pi}{b}\right]\sin\left[\frac{m\pi}{a}\right]x\cos\left[\frac{n\pi}{b}\right]y\ e^{(+j\omega t-xz)}}$$

$$\overrightarrow{Hy} = \frac{-x}{h^2}\frac{\partial \overrightarrow{Hz}}{\partial y} - \frac{j\omega\mu}{h^2}\frac{\partial \overrightarrow{Ez}}{\partial x}$$

$$\overrightarrow{Hy} = -\frac{j\omega\mu}{h^2}C\left[\frac{m\pi}{a}\right]\cos\left[\frac{m\pi}{a}\right]x\ \sin\left[\frac{n\pi}{b}\right]y\ e^{(+j\omega t-xz)}$$

TM Modes in rectangular waveguide

Depending on values of m and n, there are various modes in TM waves.

Mode representation $TM_{mn.}$

TM^{00} mode: m=0, n=0

Ex,Ey,Hx,Hy=0 so, TM_{00} cannot exist.

TM_{01} mode: m=0, n=1

Again all field component vanish so TM_{01} mode cannot exist.

TM_{10} mode: m=1, n=0

Again vanish, this mode does not exist.

TM_{11} mode: m=1, n=1.

Mode exist, become non-zero.

4 components, Ex,Ey,Hx and Hy exist with all higher mode.

Cut-off frequency of waveguides

$A = \frac{n\pi}{b}$ and $B = \frac{m\pi}{a}$

We know that

$$h^2 = A^2 + B^2 = x^2 + \omega^2 \mu\varepsilon$$

$$x^2 + \omega^2 \mu\varepsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$x = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu\varepsilon}$$

At lower frequency:

$$\omega^2 \mu\varepsilon < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

So, x:- Real and positive and equal to α, that is, wave completely attenuated and there is no phase change. Hence, wave cannot propagate.

At higher frequency,

$$\omega^2 \mu\varepsilon > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

y becomes imaginary, there will be phase change β and hence wave propagates.

At transition, x becomes zero is defined as cut-off frequency fc, (threshold frequency).

At f = fc, $x = 0$ or $\omega = 2\pi f = 2\pi fc = \omega_c$

$$0 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega_c^{\,2}\mu\varepsilon$$

$$\omega_c = \frac{1}{\sqrt{\mu\varepsilon}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2}$$

$$fc = \frac{c}{2\pi}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2}$$

$$fc = \frac{c}{2}\left[\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2}\right]$$

$$fc = \frac{C}{\lambda_c}$$

$$\lambda_{cmn} = \frac{2}{\left[\frac{m^2b^2 + n^2a^2}{a^2b^2}\right]^{1/2}}$$

$$\boxed{\lambda_{cmn} = \frac{2ab}{\sqrt{m^2b^2 + n^2a^2}}}$$

All wavelength greater than λ_c are attenuated and those less than λ_c are allowed to propagate inside the waveguide.

GUIDE WAVELENGTH, GROUP AND PHASE VELOCITY

GUIDE WAVELENGTH:-

It is defined as distance travelled by the wave in order to undergo a phase drift of 2π radians. This is shown by fig.below.

Wavelength in waveguide is different from the wavelength in free space.

Relation between free space wavelength λ_0 and cut-off wavelength λ_c by

$$\lambda_g = \frac{\lambda_0}{\sqrt{1-(\lambda_0/\lambda_c)^2}}$$

This equation is true for any mode in a waveguide of any cross-section, produced λ_c corresponds to mode and cross-section of waveguide.

If $\lambda_0 \lll \lambda_c$ then denominator =1 and $\lambda_g = \lambda_0$.

As $\lambda_0 = \lambda_c$, λ_g reaches to infinity.

When $\lambda_0 \ggg \lambda_c \Rightarrow$ λ_g will be imaginary indicates no propagation in waveguide.

PHASE VELOCITY (Vp)

Wave propagates when $\lambda_g \gg \lambda_0$ i.e.

$$V_p = \lambda_g f$$

$$C = \lambda_0 f$$

If $V_p > C$ Since, $\lambda_g > \lambda_0$ (Contradictory because $V_p < C$)

So, wavelength waveguide is length of cycle and V_p is phase velocity.

Therefore, phase velocity is defined as the rate at which the wave changes its phase in terms of guide wavelength.

That is,

$$V_p = \frac{\lambda_g}{Time} = \lambda_g . f = \frac{2\pi f . \lambda_g}{2\pi} = \frac{2\pi f}{\left(2\pi / \lambda_g\right)}$$

$$\boxed{V_p = \frac{\omega}{\beta}} \qquad \text{Since } \omega = 2\pi f \text{ and } \beta = \frac{2\pi}{\lambda_g}$$

So, no mode propagation at V_p.

$$V_p = \frac{Angular\ frequency}{phase}$$

GROUP VELOCITY

If there is modulation in carrier, the modulation evvelope actually travels at velocity slower than that of carrier alone and slower than speed of light.

Group velocity:- Velocity of modulation envelope.

This happens when modulated signal travel through waveguide.

It is defined as the rate at which the wave propagates through the waveguide and is given by-

$$V_g = \frac{d\omega}{d\beta}$$

Expression for phase velocity and group velocity

1) Expression for phase velocity (V_p):

We know that,

$$V_p = \frac{\omega}{\beta}$$

$$h^2 = A^2 + B^2 = x^2 + \omega^2 \mu\varepsilon$$
$$x = \alpha + j\beta$$
For proportion:
$$\alpha = 0 \; x = j\beta$$
$$(j\beta)^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu\varepsilon \qquad (1)$$
At $f = f_c, \; \omega = \omega_c, \; x = 0$
$$\omega_c{}^2 \mu\varepsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad (2)$$
Put value of equation (2) in equation (1)
$$(j\beta)^2 = \omega_c{}^2 \mu\varepsilon - \omega^2 \mu\varepsilon$$
$$\beta^2 = \omega^2 \mu\varepsilon - \omega_c{}^2 \mu\varepsilon$$
$$\beta = \sqrt{\omega^2 \mu\varepsilon - \omega_c{}^2 \mu\varepsilon}$$
$$\boxed{\beta = \sqrt{\mu\varepsilon}\;\sqrt{\omega^2 - \omega_c{}^2}}$$
$$V_p = \frac{\omega}{\beta}$$
$$V_p = \frac{\omega}{\sqrt{\mu\varepsilon}\;\sqrt{\omega^2 - \omega_c{}^2}} = \frac{\omega_c}{\omega\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}}$$
$$V_p = \frac{C}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}} = \frac{C}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$
$$f = \frac{C}{\lambda} \qquad f_c = \frac{C}{\lambda_c} \qquad f_0 = \frac{C}{\lambda_0}$$
$$V_p = \frac{C}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2}}$$

Expression for V_g:

$$\beta = \sqrt{\mu\varepsilon}\ \sqrt{\omega^2 - \omega_c^{\ 2}}$$

$$V_g = \frac{d\omega}{d\beta}$$

$$\frac{d\beta}{d\omega} = \frac{d\sqrt{\mu\varepsilon}\ \sqrt{\omega^2 - \omega_c^{\ 2}}}{d\omega}$$

$$\frac{d\beta}{d\omega} = \sqrt{\mu\varepsilon}\,\frac{2\omega}{2\sqrt{\omega^2 - \omega_c^{\ 2}}} = \frac{\sqrt{\mu\varepsilon}}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{1}{C\sqrt{1-\left(\frac{\lambda_0}{\lambda_c}\right)^2}}$$

$$V_g = \frac{d\omega}{d\beta} = C\sqrt{1-\left(\frac{\lambda_0}{\lambda_c}\right)^2}$$

$So,$

$$V_p.V_g = \frac{C}{\sqrt{1-\left(\frac{\lambda_0}{\lambda_c}\right)^2}}.C\sqrt{1-\left(\frac{\lambda_0}{\lambda_c}\right)^2}$$

$$\boxed{V_p.V_g = C^2}$$

Relation between λ_g, λ_0 and λ_c:

$$V_p = \lambda_g.f = \frac{\lambda_g C}{\lambda_0}$$

$$V_p = \frac{C}{\sqrt{1-\left(\frac{\lambda_0}{\lambda_c}\right)^2}}$$

$$\lambda_g = \frac{\lambda_0}{\sqrt{1-\left(\frac{\lambda_0}{\lambda_c}\right)^2}}$$

TM MODES IN RECTANGULAR WAVEGUIDES

1) TM_{11} mode:
   Minimum possible mode
   m=1,n=1

$$\lambda_{cmn} = \frac{2ab}{\sqrt{m^2b^2 + n^2a^2}}$$

$$m = n = 1$$

$$\lambda_{cmn} = \frac{2ab}{\sqrt{a^2 + b^2}}$$

2) TM_{12} mode:
   m=1, n=2

$$\lambda_{c12} = \frac{2ab}{\sqrt{b^2 + 4a^2}}$$

3) TM_{21} mode:
   m=2,n=1

$$\lambda_{c21} = \frac{2ab}{\sqrt{4b^2 + a^2}}$$

PROPAGATION OF TE WAVES IN RECTANGULAR WAVEGUIDE

$Ez = 0,\ \mathrm{Hz} \neq 0$

Helmholtz wave equation

n:

$$\nabla^2 \overrightarrow{Hz} = -\omega^2 \mu\varepsilon \overrightarrow{Hz}$$

$$\frac{\partial^2 \overrightarrow{Hz}}{\partial x^2} + \frac{\partial^2 \overrightarrow{Hz}}{\partial y^2} + \frac{\partial^2 \overrightarrow{Hz}}{\partial z^2} = -\omega^2 \mu\varepsilon \overrightarrow{Hz}$$

$$\frac{\partial^2 \overrightarrow{Hz}}{\partial x^2} + \frac{\partial^2 \overrightarrow{Hz}}{\partial y^2} + x^2 \overrightarrow{Hz} = -\omega^2 \mu\varepsilon \overrightarrow{Hz}$$

$$\frac{\partial^2 \overrightarrow{Hz}}{\partial x^2} + \frac{\partial^2 \overrightarrow{Hz}}{\partial y^2} + (x^2 + \omega^2 \mu\varepsilon) \overrightarrow{Hz} = 0$$

Let solution of equation:

$Hz = XY$

$X =$ pure function of x

$Y =$ pure function of y

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} + (x^2 + \omega^2 \mu\varepsilon) XY = 0$$

Divide by XY

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} + h^2 = 0$$

$\frac{1}{X} \frac{\partial^2 X}{\partial x^2}$ = pure function of X Let = $-\mathrm{B}^2$

$\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2}$ = pure function of Y Let = $-\mathrm{A}^2$

$$-\mathrm{B}^2 - \mathrm{A}^2 + h^2 = 0$$

$$\boxed{h^2 = \mathrm{A}^2 + \mathrm{B}^2}$$

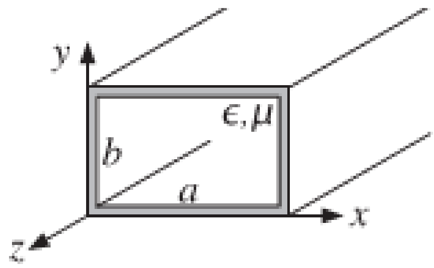
So, X and Y by separation of variable method:

$$X = C_1 \cos\ Bx + C_2\ \sin\ Bx$$

$$Y = C_3\ \cos\ Ay + C_4\ \sin\ Ay$$

$$Hz = XY = (C_1 \cos\ Bx + C_2\ \sin\ Bx)(C_3\ \cos\ Ay + C_4\ \sin\ Ay)$$

BOUNDARY CONDITIONS:



Here since we are considering TE wave:

Ez=0 but we have components along x and y direction.

Ex=0 cell bottom and top walls of waveguide.

Ey=0 all side (left and right) wall of waveguide.

I Boundary condition-

$\overrightarrow{Ex} = 0$ y=0 & $x \to 0$ to a

II Boundary Condition: (LHS plane)

$\overrightarrow{Ey} = 0 \; x = 0 \quad y \to 0$ to b

III Boundary Condition: (Top plane)

$\overrightarrow{Ex} = 0 \; y = b \quad x \to 0 \; to \; a$

IV Boundary Condition: (RHS plane)

$\overrightarrow{Ey} = 0 \; x = a \quad y \to 0$ to b

1) Substituting I Boundary condition in

$$\overrightarrow{Ex} = \frac{-x}{h^2}\frac{\partial \overrightarrow{Ez}}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial \overrightarrow{Hz}}{\partial y}$$

$$\overrightarrow{Ex} = \frac{-x}{h^2}\frac{\partial \overrightarrow{Ez}}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial \overrightarrow{Hz}}{\partial y}$$

$$0 = -\frac{j\omega\mu}{h^2}\frac{\partial}{\partial y}(C_1\cos\ Bx + C_2\ \sin\ Bx)(C_3\ \cos\ Ay + C_4\ \sin\ Ay)$$

$$(C_1\cos\ Bx + C_2\ \sin\ Bx)(C_3\ Ac\text{os}\ Ay + C_4\text{A}\sin\ Ay) = 0$$

Put I B.C:

$$y = 0$$

$$(C_1\cos\ Bx + C_2\ \sin\ Bx)C_4 A = 0$$

$$A \neq 0,\ C_4 = 0$$

$$Hz = (C_1\cos\ Bx + C_2\sin\ Bx)C_3\ c\text{os}\ Ay \qquad (2)$$

II Boundary Condition

$$\overrightarrow{Ey} = \frac{-x}{h^2}\frac{\partial \overrightarrow{Ez}}{\partial x} + \frac{j\omega\mu}{h^2}\frac{\partial \overrightarrow{Hz}}{\partial y}$$

Put $\overrightarrow{\text{Hz}}$ from (2) equation:

$$\overrightarrow{Ey} = \frac{j\omega\mu}{h^2}\frac{\partial}{\partial y}\left[C_3\ \cos\ Ay(C_1\cos\ Bx + C_2\ \sin\ Bx)\right]$$

$$\overrightarrow{Ey} = \frac{x}{h^2}\left[C_3\ \cos\ Ay(-BC_1\cos\ Bx + BC_2\sin\ Bx)\right]$$

Apply II B.C: $(x = 0)$

$$\frac{j\omega\mu}{h^2}(C_3\ \cos\ Ay.BC_2) = 0$$

$$B \neq 0, C_3\ \neq 0, C_2 = 0$$

Apply C_2 in equation (2)

$$\boxed{\overrightarrow{Hz} = C_1 C_3\ \cos\ Bx\cos\ Ay} \qquad (3)$$

III BOUNDARY CONDITION

$$\overrightarrow{Ex} = \frac{-x}{h^2}\frac{\partial \overrightarrow{Ez}}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial \overrightarrow{Hz}}{\partial y}$$

$$\overrightarrow{Ex} = \frac{j\omega\mu}{h^2}\frac{\partial}{\partial y}C_1C_3 \cos Bx \cos Ay$$

$$\overrightarrow{Ex} = \frac{j\omega\mu}{h^2}AC_1C_3 \cos Bx \sin Ay$$

Apply III B.C

$$\frac{j\omega\mu}{h^2}AC_1C_3 \cos Bx \sin Ay = 0$$

$$\frac{j\omega\mu}{h^2}AC_1C_3 \cos Bx \sin Ab = 0$$

$C_1C_3 \cos Bx \neq 0$

sin Ab=0

Ab=nπ

$$\boxed{A = \frac{n\pi}{b}}$$

IV B.C

$$\overrightarrow{Ey} = \frac{-x}{h^2}\frac{\partial \overrightarrow{Ez}}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial \overrightarrow{Hz}}{\partial y}$$

$$\overrightarrow{Ez} = 0$$

$$E_g = \frac{j\omega\mu}{h^2}\frac{\partial \overrightarrow{Hz}}{\partial x}$$

$$E_g = \frac{j\omega\mu}{h^2}\frac{\partial}{\partial x}C_1C_3 \cos Bx \cos Ay$$

$$E_g = -\frac{j\omega\mu}{h^2}C_1C_3\, B \sin Bx \cos Ay$$

Apply IV B.C

$$0 = -\frac{j\omega\mu}{h^2}C_1C_3\, B \sin Ba \cos Ay$$

$C_1C_3 \cos Ay \neq 0$

sin Ba=0

Ba=nπ

$$B = \frac{m\pi}{a}$$

Put A and B in equation (3)

$$\boxed{\overrightarrow{Hz} = C_1C_3\ \cos\left(\frac{m\pi}{a}\right)\ x\cos\left(\frac{n\pi}{b}\right)y.e^{j\omega t - xz}}$$

So,

$$\overrightarrow{Ex} = \frac{-x}{h^2}\frac{\partial\overrightarrow{Ez}}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial\overrightarrow{Hz}}{\partial y}$$

$$\overrightarrow{Ex} = -\frac{j\omega\mu}{h^2}\frac{\partial}{\partial y}C_1C_3\ \cos\left(\frac{m\pi}{a}\right)\ x\cos\left(\frac{n\pi}{b}\right)y.e^{j\omega t - xz}$$

$$\boxed{\overrightarrow{Ex} = +\frac{j\omega\mu}{h^2}C\left(\frac{n\pi}{b}\right)\cos\left(\frac{m\pi}{a}\right)\ x\sin\left(\frac{n\pi}{b}\right)y.e^{j\omega t - xz}}$$

$$\overrightarrow{Ey} = \frac{-x}{h^2}\frac{\partial\overrightarrow{Ez}}{\partial x} + \frac{j\omega\mu}{h^2}\frac{\partial\overrightarrow{Hz}}{\partial y}$$

$$\overrightarrow{Ey} = \frac{j\omega\mu}{h^2}\frac{\partial}{\partial y}\left[C\cos\left(\frac{m\pi}{a}\right)x\cos\left(\frac{n\pi}{b}\right)y.e^{j\omega\varepsilon - xz}\right]$$

$$\boxed{\overrightarrow{Ey} = -\frac{j\omega\mu}{h^2}\left[C\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi}{a}\right)x\cos\left(\frac{n\pi}{b}\right)y.e^{j\omega t - xz}\right]}$$

$$\overrightarrow{Hx} = \frac{-x}{h^2}\frac{\partial\overrightarrow{Hz}}{\partial x} + \frac{j\omega\mu}{h^2}\frac{\partial\overrightarrow{Ez}}{\partial y}$$

$$\overrightarrow{Hx} = \frac{-x}{h^2}\frac{\partial}{\partial x}\left[C\cos\left(\frac{m\pi}{a}\right)x\cos\left(\frac{n\pi}{b}\right)y.e^{j\omega t - xz}\right]$$

$$\boxed{\overrightarrow{Hx} = \frac{+x}{h^2}C\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi}{a}\right)\ x\cos\left(\frac{n\pi}{b}\right)y.e^{j\omega t - xz}}$$

and

$$\overrightarrow{Hy} = \frac{-x}{h^2}\frac{\partial\overrightarrow{Hz}}{\partial y} - \frac{j\omega\mu}{h^2}\frac{\partial\overrightarrow{Ez}}{\partial y}$$

$$\boxed{\overrightarrow{Hy} = \frac{+x}{h^2}C\left(\frac{n\pi}{b}\right)\cos\left(\frac{m\pi}{a}\right)x\sin\left(\frac{n\pi}{b}\right)\ y.e^{j\omega t - xz}}$$

TE Modes:

$$\overrightarrow{Ex} = +\frac{j\omega\mu}{h^2} C\left(\frac{n\pi}{b}\right)\cos\left(\frac{m\pi}{a}\right) x \sin\left(\frac{n\pi}{b}\right) y.e^{j\omega t - xz} \quad (4)$$

$$\overrightarrow{Ey} = -\frac{j\omega\mu}{h^2}\left[C\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi}{a}\right) x \cos\left(\frac{n\pi}{b}\right) y.e^{j\omega t - xz}\right] (5)$$

$$\overrightarrow{Hx} = \frac{+x}{h^2} C\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi}{a}\right) x \cos\left(\frac{n\pi}{b}\right) y.e^{j\omega t - xz} \qquad (6)$$

$$\overrightarrow{Hy} = \frac{+x}{h^2} C\left(\frac{n\pi}{b}\right)\cos\left(\frac{m\pi}{a}\right) x \sin\left(\frac{n\pi}{b}\right) y.e^{j\omega t - xz} \qquad (7)$$

TE MODES IN RECTANGULAR WAVEGUIDE

1) TE_{00} mode:

Minimum possible mode

m=0 ,n=0

Ex=Ey=Hx=Hy=0 (Not exist)

2) TE_{01} mode:

m=0, n=1

Ex= Hy $\neq 0$

Ey=Hx=0

Exist

3) TE_{10} mode:

m=1,n=0

Ex= Hy=0

Ey=Hx $\neq 0$

Exist.

4) TE_{11} mode:

Ex=Ey=Hx=Hy $\neq 0$

Exist and even higher mode also exist.

DOMINANT MODE:

It is that mode for which cut-off wavelength (λ_c) is max. value.

$$\lambda_{cmn} = \frac{2ab}{\sqrt{m^2b^2 + n^2a^2}}$$

TE_{01} mode:

$$\lambda_{c01} = \frac{2ab}{\sqrt{a^2}}$$

TE_{10} mode:

$$\lambda_{c10} = \frac{2ab}{\sqrt{b^2}}$$

TE_{11} mode:

$$\lambda_{c11} = \frac{2ab}{\sqrt{a^2 + b^2}}$$

Out of these λ_{c10} have maximum value because a is larger dimension. So, TE_{10} mode is dominant mode in rectangular waveguide.

Other Expressions:

$$V_p = \lambda_g . f = \frac{\lambda_g C}{\lambda_0}$$

$$V_p = \frac{C}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2}}$$

$$\lambda_g = \frac{\lambda_0}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2}}$$

DOMINANT MODE AND DEGENERATE MODE IN RECTANGULAR WAVEGUIDE

Walls of waveguide is perfect conductor.

Therefore, Boundary condition required: Electric field normal or perpendicular and magnetic field tangential i.e. parallel to waveguide.

Zero subject can exist in TE mode but not in TM mode.

TE_{10}, TE_{01} and TE_{20} etc modes can exist in rectangular waveguide but also TM_{11}, TM_{12}, TM_{21} etc. mode can exist.

Also, cut-off frequency relationship shows that physical size of waveguide determines propagation of modes depending on values of m and n.

Minimum cut-off frequency for rectangular waveguide is obtained for dimension $a>b$ for $m=1$ and $n=0$ i.e. TE_{10} mode is dominant mode for rectangular waveguide. (for $a>b$)

Other higher modes having same cut-off frequency are called degenerate mode.

For rectangular waveguide TM_{mn}/TE_{mn} modes for which both $m\neq0$ and $n\neq0$ will always be degenerate modes.

For square waveguide, $a=b$, all TE_{pq}, TE_{qp}, TM_{pq}, TM_{qp} modes are degenerate modes.

CIRCULAR WAVEGUIDES

The circular waveguide is occasionally used as an alternative to the rectangular waveguide. Like other waveguides constructed from a single, enclosed conductor, the circular waveguide supports transverse electric (TE) and transverse magnetic (TM) modes. These modes have a cutoff frequency, below which electromagnetic energy is severely attenuated. Circular waveguide's round cross section makes it easy to machine, and it is often used to feed conical horns. Further, the TE0n modes of circular waveguide have very low attenuation. A disadvantage of circular waveguide is its limited dominant mode bandwidth, which, compared to rectangular waveguide's maximum bandwidth of 2–1, is only 1.3. In addition, the polarization of the dominant mode is arbitrary, so that discontinuities can easily excite unwanted cross-polarized components.

TRANSVERSE ELECTRIC (TEZ) MODES The transverse electric (TEz) modes can be derived by letting the vector potential A and F be equal to

$$\mathbf{A} = 0 \qquad (1a)$$

$$\mathbf{F} = \hat{a}_z F_z(\rho, \phi, z) \qquad (1b)$$

The vector potential F must satisfy the vector wave equation, which reduces the F of (1b) to

$$\nabla^2 F_z(\rho, \phi, z) + \beta^2 F_z(\rho, \phi, z) = 0 \qquad (2)$$

When expanded in cylindrical coordinates, (2) reduces to

$$\frac{\partial^2 F_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 F_z}{\partial \phi^2} + \frac{\partial^2 F_z}{\partial z^2} + \beta^2 F_z = 0 \qquad (3)$$

whose solution for the geometry of Fig. 1 is of the form

$$F_z(\rho,\phi,z)=[A_1J_m(\beta_\rho\rho)+B_1Y_m(\beta_\rho\rho)] \times[C_2\cos(m\phi)+D_2\sin(m\phi)] \times[A_3e^{-j\beta_z z}+B_3e^{+j\beta_z z}] \quad (4a)$$

where

$$\beta_\rho^2+\beta_z^2=\beta^2 \quad (4b)$$

The constants A1, B1, C2, D2, A3, B3, m, β_ρ, and β_z can be found using the boundary conditions of

$$E_\phi(\rho=a,\phi,z)=0 \quad (5a)$$

The fields must be finite everywhere (5b)

The fields must repeat every 2π radians in ϕ (5c)

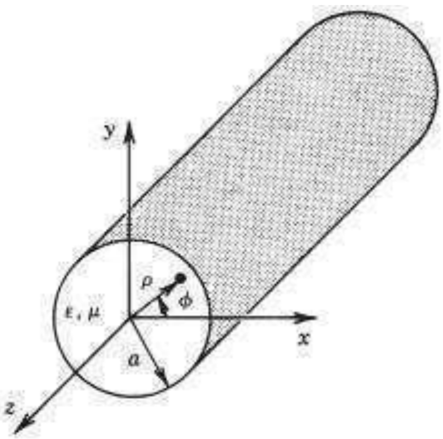


Figure 1. Cylindrical waveguide of circular cross section.

According to (5b), B1= 0 since Ym(ρ=0)=∞. In addition, according to (5c)

$$m=0,1,2,3,\ldots \quad (6)$$

Consider waves that propagate only in the +z direction. Then (4) reduces to

$$F_z^+(\rho,\phi,z)=A_{mn}J_m(\beta_\rho\rho)[C_2\cos(m\phi)+D_2\sin(m\phi)]e^{-j\beta_z z} \quad (7)$$

From Eq. (7), the electric field component of E_ϕ^+ can be written as

$$E_\phi^+ = \frac{1}{\varepsilon}\frac{\partial F_z^+}{\partial \rho}$$
$$= \beta_\rho \frac{A_{mn}}{\varepsilon} J'_m(\beta_\rho \rho)[C_2 \cos(m\phi) + D_2 \sin(m\phi)]e^{-j\beta_z z} \quad (8a)$$

where

$$' = \frac{\partial}{\partial(\beta_\rho \rho)} \quad (8b)$$

Applying the boundary condition of (5a) in (8a), we then have that

$$E_\phi^+(\rho = a, \phi, z) = \beta_\rho \frac{A_{mn}}{\varepsilon} J'_m(\beta_\rho a)[C_2 \cos(m\phi) + D_2 \sin(m\phi)]e^{-j\beta_z z} = 0 \quad (9)$$

which is satisfied only provided that

$$J'_m(\beta_\rho a) = 0 \Rightarrow \beta_\rho a = \chi'_{mn} \Rightarrow \beta_\rho = \frac{\chi'_{mn}}{a} \quad (10)$$

In (10) χ_{mn} represents the nth zero (n=1, 2, 3,…) of the derivative of the Bessel function J_m of the first kind of order m(m = 0, 1, 2, 3,,,,). An abbreviated list of the zeros χ_{mn} of the derivative Jm0 of the Bessel function Jm is found in Table 1. The smallest value of w0mn is 1.8412 (m = 1, n = 1).

Using (4b) and (10), βz of the mn mode can be written as

$$(\beta_z)_{mn} = \begin{cases} \sqrt{\beta^2 - \beta_\rho^2} = \sqrt{\beta^2 - \left(\frac{\chi'_{mn}}{a}\right)^2} \\ \text{when } \beta > \beta_\rho = \dfrac{\chi'_{mn}}{a} \end{cases} \tag{11a}$$

$$(\beta_z)_{mn} = \begin{cases} 0 \quad \text{when } \beta = \beta_c = \beta_\rho = \dfrac{\chi'_{mn}}{a} \end{cases} \tag{11b}$$

$$(\beta_z)_{mn} = \begin{cases} -j\sqrt{\beta_\rho^2 - \beta^2} = -j\sqrt{\left(\dfrac{\chi'_{mn}}{a}\right)^2 - \beta^2} \\ \text{when } \beta < \beta_\rho = \dfrac{\chi'_{mn}}{a} \end{cases} \tag{11c}$$

Cutoff is defined when $(\beta_z)_{mn} = 0$. Thus, according to (11b)

$$\beta_c = \omega_c\sqrt{\mu\varepsilon} = 2\pi f_c\sqrt{\mu\varepsilon} = \beta_\rho = \frac{\chi'_{mn}}{a} \tag{12a}$$

$$(f_c)_{mn} = \frac{\chi'_{mn}}{2\pi a\sqrt{\mu\varepsilon}} \tag{12b}$$

By using (12a) and (12b), we can write (11a)–(11c) as

$$(\beta_z)_{mn} = \begin{cases} \sqrt{\beta^2 - \beta_\rho^2} = \beta\sqrt{1 - \left(\dfrac{\beta_\rho}{\beta}\right)^2} = \beta\sqrt{1 - \left(\dfrac{\beta_c}{\beta}\right)^2} \\ = \beta\sqrt{1 - \left(\dfrac{\chi'_{mn}}{\beta a}\right)^2} = \beta\sqrt{1 - \left(\dfrac{f_c}{f}\right)^2} \\ \text{when } f > f_c = (f_c)_{mn} \end{cases} \tag{13a}$$

$$(\beta_z)_{mn} = \begin{cases} 0 \quad \text{when } f = f_c = (f_c)_{mn} \end{cases} \tag{13b}$$

$$(\beta_z)_{mn} = \begin{cases} -j\sqrt{\beta_\rho^2 - \beta^2} = -j\beta\sqrt{\left(\frac{\beta_\rho}{\beta}\right)^2 - 1} = -j\beta\sqrt{\left(\frac{\beta_c}{\beta}\right)^2 - 1} \\ = -j\beta\sqrt{\left(\frac{\chi_{mn}}{\beta a}\right)^2 - 1} = -j\beta\sqrt{\left(\frac{f_c}{f}\right)^2 - 1} \\ \text{when } f < f_c = (f_c)_{mn} \end{cases} \quad (13c)$$

The guide wavelength λ_g is defined as

$$(\lambda_g)_{mn} = \frac{2\pi}{(\beta_z)_{mn}} \quad (14a)$$

which, according to (13a) and (13b), can be written as

$$(\lambda_g)_{mn} = \begin{cases} \dfrac{2\pi}{\beta\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = \dfrac{\lambda}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \quad \text{when } f > f_c = (f_c)_{mn} \end{cases}$$

(14b)

$$(\lambda_g)_{mn} = \{\infty \quad \text{when } f = (f_c)_{mn} \quad (14c)$$

In (14b) l is the wavelength of the wave in an infinite medium of the kind that exists inside the waveguide. There is no definition of the wavelength below cutoff since the wave is exponentially decaying and there is no repetition of its waveform.

According to (12b) and the values of χ_{mn} in Table 1, the order (lower to higher cutoff frequencies) in which the TE_z^{mn} modes occur is TEz11, TEz21, TEz01, and so on. It should be noted that for a circular waveguide, the order in which the TEz^{mn} modes occur does not change, and the bandwidth between modes is also fixed. For example, the bandwidth of the first single-mode TEz^{11} operation is 3.042/1.8412= 1.6588 : 1, which is less than 2 : 1. This bandwidth is fixed and cannot be varied. A change in the radius only varies, by the same amount, the absolute values of the cutoff frequencies of all the modes but does not alter their order or relative bandwidth.

The electric and magnetic field components can be written from Eq. (7) as

$$E_\rho^+ = -\frac{1}{\varepsilon\rho}\frac{\partial F_z^+}{\partial \phi} = -A_{mn}\frac{m}{\varepsilon\rho}J_m(\beta_\rho\rho)[-C_2\sin(m\phi) + D_2\cos(m\phi)]e^{-j\beta_z z} \tag{15a}$$

$$E_\phi^+ = \frac{1}{\varepsilon}\frac{\partial F_z^+}{\partial \rho} = A_{mn}\frac{\beta_\rho}{\varepsilon}J'_m(\beta_\rho\rho)[C_2\cos(m\phi) + D_2\sin(m\phi)]e^{-j\beta_z z} \tag{15b}$$

$$E_z^+ = 0 \tag{15c}$$

$$H_\rho^+ = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z^+}{\partial\rho\partial z} = -A_{mn}\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon}J'_m(\beta_\rho\rho)[C_2\cos(m\phi) + D_2\sin(m\phi)]e^{-j\beta_z z} \tag{15d}$$

$$H_\phi^+ = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 F_z^+}{\partial\phi\partial z} = -A_{mn}\frac{m\beta_z}{\omega\mu\varepsilon}\frac{1}{\rho}J_m(\beta_\rho\rho) \times [-C_2\sin(m\phi) + D_2\cos(m\phi)]e^{-j\beta_z z} \tag{15e}$$

$$H_z^+ = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)F_z^+ = -jA_{mn}\frac{\beta_\rho^2}{\omega\mu\varepsilon}J_m(\beta_\rho\rho) \times [C_2\cos(m\phi) + D_2\sin(m\phi)]e^{-j\beta_z z} \tag{15f}$$

where

$$' = \frac{\partial}{\partial(\beta_\rho\rho)} \tag{15g}$$

$$Z_{mn}^h = (Z_w^{+z})_{mn}^{\mathrm{TE}} = \frac{E_\rho^+}{H_\phi^+} = -\frac{E_\phi^+}{H_\rho^+} = \frac{\omega\mu}{(\beta_z)_{mn}} \tag{16a}$$

$$Z^h_{mn} = (Z_w^{+z})^{TE}_{mn} = \begin{cases} \dfrac{\omega\mu}{\beta\sqrt{1-\left(\dfrac{f_c}{f}\right)^2}} = \dfrac{\sqrt{\dfrac{\mu}{\varepsilon}}}{\sqrt{1-\left(\dfrac{f_c}{f}\right)^2}} = \dfrac{\eta}{\sqrt{1-\left(\dfrac{f_c}{f}\right)^2}} \\ \text{when } f > f_c = (f_c)_{mn} \end{cases} \tag{16b}$$

$$Z^h_{mn} = (Z_w^{+z})^{TE}_{mn} = \left\{ \frac{\omega\mu}{0} = \infty \quad \text{when } f = f_c = (f_c)_{mn} \right. \tag{16c}$$

$$Z^h_{mn} = (Z_w^{+z})^{TE}_{mn} = \begin{cases} \dfrac{\omega\mu}{-j\beta\sqrt{\left(\dfrac{f_c}{f}\right)^2-1}} = +j\dfrac{\sqrt{\dfrac{\mu}{\varepsilon}}}{\sqrt{\left(\dfrac{f_c}{f}\right)^2-1}} = +j\dfrac{\eta}{\sqrt{\left(\dfrac{f_c}{f}\right)^2-1}} \\ \text{when } f < f_c = (f_c)_{mn} \end{cases} \tag{16d}$$

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